

# Forward-Backward Splitting with Bregman Distances

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## Abstract

We propose a forward-backward splitting algorithm based on Bregman distances for composite minimization problems in general reflexive Banach spaces. The convergence is established using the notion of variable quasi-Bregman monotone sequences. Various examples are discussed, including some in Euclidean spaces, where new algorithms are obtained.

**Key words.** Banach space, Bregman distance, forward-backward algorithm, Legendre function, multivariate minimization, variable quasi-Bregman monotonicity.

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# 1 Introduction

We consider the following composite convex minimization problem.

**Problem 1.1** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be reflexive real Banach spaces, let  $\varphi \in \Gamma_0(\mathcal{X})$ , let  $\psi \in \Gamma_0(\mathcal{Y})$  be Gâteaux differentiable on  $\text{int dom } \psi \neq \emptyset$ , and let  $L: \mathcal{X} \rightarrow \mathcal{Y}$  be a bounded linear operator. The problem is to

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad \varphi(x) + \psi(Lx). \quad (1.1)$$

The set of solutions to (1.1) is denoted by  $\mathcal{S}$ .

A particular instance of (1.1) is when  $\psi = D^g(\cdot, r)$ , where  $g \in \Gamma_0(\mathcal{Y})$  is Gâteaux differentiable on  $\text{int dom } g \ni r$ , i.e.,

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad \varphi(x) + D^g(Lx, r). \quad (1.2)$$

This model provides a framework for many problems arising in applied mathematics. For instance, when  $\mathcal{X}$  and  $\mathcal{Y}$  are Euclidean spaces and  $g$  is Boltzmann-Shannon entropy, it captures many problems in information theory and signal recovery [10]. Besides, the nearness matrix problem [20] and the log-determinant minimization problem [14] can be also regarded as special cases of (1.2).

An objective is constructing effective splitting methods, i.e, the methods that activate each function in the model separately, to solve Problem 1.1 (see [18] for more discussions). It was shown in [18] that if  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces and if  $\psi$  possess a  $\beta^{-1}$ -Lipschitz continuous gradient for some  $\beta \in ]0, +\infty[$ , then Problem 1.1 can be solved by the standard forward-backward algorithm

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \text{prox}_{\gamma_n \varphi}(x_n - \gamma_n L^*(\nabla \psi(Lx_n))), \quad \text{where} \quad 0 < \gamma_n < 2\beta. \quad (1.3)$$

Here,  $(\text{prox}_{\gamma_n \varphi})_{n \in \mathbb{N}}$  are the Moreau proximity operators [24]. However, many problems in applications do not conform to these hypotheses, for example when  $\mathcal{X}$  and  $\mathcal{Y}$  are Euclidean spaces and  $\psi$  is Boltzmann-Shannon entropy. This type of functions appears in many problems in image and signal processing, in statistics, and in machine learning [2, 13, 14, 21, 22, 23]. Another difficulty in the implementation of (1.3) is that the operators  $(\text{prox}_{\gamma_n \varphi})_{n \in \mathbb{N}}$  are not always easy to evaluate. The objective of the present paper is to propose a version of the forward-backward splitting algorithm to solve Problem 1.1, which is so far limited to Hilbert spaces, in the general framework of reflexive real Banach spaces. This algorithm, which employs Bregman distance-based proximity operators, provides new algorithms in the framework of Euclidean spaces, which are, in some instances, more favorable than the standard forward-backward splitting algorithm. This framework can be applied in the case when  $\psi$  is not everywhere differentiable and in some instances, it requires less efforts in the computation of proximity operators than the classical framework. This paper revolves around the following definitions.

**Definition 1.2** [5, 6] Let  $\mathcal{X}$  be a reflexive real Banach space, let  $\mathcal{X}^*$  be the topological dual space of  $\mathcal{X}$ , let  $\langle \cdot, \cdot \rangle$  be the duality pairing between  $\mathcal{X}$  and  $\mathcal{X}^*$ , let  $f: \mathcal{X} \rightarrow ]-\infty, +\infty]$  be a lower semicontinuous convex function that is Gâteaux differentiable on  $\text{int dom } f \neq \emptyset$ , let  $f^*: \mathcal{X}^* \rightarrow ]-\infty, +\infty]: x^* \mapsto \sup_{x \in \mathcal{X}} (\langle x, x^* \rangle - f(x))$  be conjugate of  $f$ , and let

$$\partial f: \mathcal{X} \rightarrow 2^{\mathcal{X}^*}: x \mapsto \{x^* \in \mathcal{X}^* \mid (\forall y \in \mathcal{X}) \langle y - x, x^* \rangle + f(x) \leq f(y)\}, \quad (1.4)$$

be the Moreau subdifferential of  $f$ . The *Bregman distance* associated with  $f$  is

$$D^f : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty]$$

$$(x, y) \mapsto \begin{cases} f(x) - f(y) - \langle x - y, \nabla f(y) \rangle, & \text{if } y \in \text{int dom } f; \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.5)$$

In addition,  $f$  is a *Legendre function* if it is *essentially smooth* in the sense that  $\partial f$  is both locally bounded and single-valued on its domain, and *essentially strictly convex* in the sense that  $\partial f^*$  is locally bounded on its domain and  $f$  is strictly convex on every convex subset of  $\text{dom } \partial f$ . Let  $C$  be a closed convex subset of  $\mathcal{X}$  such that  $C \cap \text{int dom } f \neq \emptyset$ . The *Bregman projector* onto  $C$  induced by  $f$  is

$$P_C^f : \text{int dom } f \rightarrow C \cap \text{int dom } f$$

$$y \mapsto \underset{x \in C}{\operatorname{argmin}} D^f(x, y), \quad (1.6)$$

and the  $D^f$ -distance to  $C$  is the function

$$D_C^f : \mathcal{X} \rightarrow [0, +\infty]$$

$$y \mapsto \inf D^f(C, y). \quad (1.7)$$

The paper is organized as follows. In Section 2, we provide some preliminary results. We present the forward-backward splitting algorithm in reflexive Banach spaces in Section 3. Section 4 is devoted to an application of our result to multivariate minimization problem together with examples.

**Notation and background.** The norm of a Banach space is denoted by  $\|\cdot\|$ . The symbols  $\rightharpoonup$  and  $\rightarrow$  represent respectively weak and strong convergence. The set of weak sequential cluster points of a sequence  $(x_n)_{n \in \mathbb{N}}$  is denoted by  $\mathfrak{W}(x_n)_{n \in \mathbb{N}}$ . Let  $M : \mathcal{X} \rightarrow 2^{\mathcal{X}^*}$ . The domain of  $M$  is  $\text{dom } M = \{x \in \mathcal{X} \mid Mx \neq \emptyset\}$  and the range of  $M$  is  $\text{ran } M = \{x^* \in \mathcal{X}^* \mid (\exists x \in \mathcal{X}) x^* \in Mx\}$ . Let  $f : \mathcal{X} \rightarrow ]-\infty, +\infty]$ . Then  $f$  is cofinite if  $\text{dom } f^* = \mathcal{X}^*$ , is coercive if  $\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$ , is supercoercive if  $\lim_{\|x\| \rightarrow +\infty} f(x)/\|x\| = +\infty$ , and is uniformly convex at  $x \in \text{dom } f$  if there exists an increasing function  $\phi : [0, +\infty[ \rightarrow [0, +\infty]$  that vanishes only at 0 such that

$$(\forall y \in \text{dom } f)(\forall \alpha \in ]0, 1[) \quad f(\alpha x + (1 - \alpha)y) + \alpha(1 - \alpha)\phi(\|x - y\|) \leq \alpha f(x) + (1 - \alpha)f(y). \quad (1.8)$$

Denote by  $\Gamma_0(\mathcal{X})$  the class of all lower semicontinuous convex functions  $f : \mathcal{X} \rightarrow ]-\infty, +\infty]$  such that  $\text{dom } f = \{x \in \mathcal{X} \mid f(x) < +\infty\} \neq \emptyset$ . Let  $f \in \Gamma_0(\mathcal{X})$ . The set of global minimizers of  $f$  is denoted by  $\operatorname{Argmin} f$ . Finally,  $\ell_+^1(\mathbb{N})$  is the set of all summable sequences in  $[0, +\infty[$ .

## 2 Preliminary results

First, we recall the following definitions and results.

**Definition 2.1** [25] Let  $\mathcal{X}$  be a reflexive real Banach space and let  $f \in \Gamma_0(\mathcal{X})$  be Gâteaux differentiable on  $\text{int dom } f \neq \emptyset$ . Then

$$\mathcal{F}(f) = \{g \in \Gamma_0(\mathcal{X}) \mid g \text{ is Gâteaux differentiable on } \text{int dom } g = \text{int dom } f\}. \quad (2.1)$$

Moreover, if  $g_1$  and  $g_2$  are in  $\mathcal{F}(f)$ , then

$$g_1 \succcurlyeq g_2 \iff (\forall x \in \text{dom } f)(\forall y \in \text{int dom } f) \quad D^{g_1}(x, y) \geq D^{g_2}(x, y). \quad (2.2)$$

For every  $\alpha \in [0, +\infty[$ , set

$$\mathcal{P}_\alpha(f) = \{g \in \mathcal{F}(f) \mid g \succcurlyeq \alpha f\}. \quad (2.3)$$

**Definition 2.2** [25] Let  $\mathcal{X}$  be a reflexive real Banach space, let  $f \in \Gamma_0(\mathcal{X})$  be Gâteaux differentiable on  $\text{int dom } f \neq \emptyset$ , let  $(f_n)_{n \in \mathbb{N}}$  be in  $\mathcal{F}(f)$ , let  $(x_n)_{n \in \mathbb{N}} \in (\text{int dom } f)^\mathbb{N}$ , and let  $C \subset \mathcal{X}$  be such that  $C \cap \text{dom } f \neq \emptyset$ . Then  $(x_n)_{n \in \mathbb{N}}$  is:

(i) *quasi-Bregman monotone* with respect to  $C$  relative to  $(f_n)_{n \in \mathbb{N}}$  if

$$\begin{aligned} &(\exists (\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N}))(\forall x \in C \cap \text{dom } f)(\exists (\varepsilon_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N}))(\forall n \in \mathbb{N}) \\ &D^{f_{n+1}}(x, x_{n+1}) \leq (1 + \eta_n)D^{f_n}(x, x_n) + \varepsilon_n; \end{aligned} \quad (2.4)$$

(ii) *stationarily quasi-Bregman monotone* with respect to  $C$  relative to  $(f_n)_{n \in \mathbb{N}}$  if

$$\begin{aligned} &(\exists (\varepsilon_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N}))(\exists (\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N}))(\forall x \in C \cap \text{dom } f)(\forall n \in \mathbb{N}) \\ &D^{f_{n+1}}(x, x_{n+1}) \leq (1 + \eta_n)D^{f_n}(x, x_n) + \varepsilon_n. \end{aligned} \quad (2.5)$$

**Condition 2.3** [6, Condition 4.4] Let  $\mathcal{X}$  be a reflexive real Banach space and let  $f \in \Gamma_0(\mathcal{X})$  be Gâteaux differentiable on  $\text{int dom } f \neq \emptyset$ . For every bounded sequences  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  in  $\text{int dom } f$ ,

$$D^f(x_n, y_n) \rightarrow 0 \implies x_n - y_n \rightarrow 0. \quad (2.6)$$

**Proposition 2.4** [25] Let  $\mathcal{X}$  be a reflexive real Banach space, let  $f \in \Gamma_0(\mathcal{X})$  be Gâteaux differentiable on  $\text{int dom } f \neq \emptyset$ , let  $\alpha \in ]0, +\infty[$ , let  $(f_n)_{n \in \mathbb{N}}$  be in  $\mathcal{P}_\alpha(f)$ , let  $(x_n)_{n \in \mathbb{N}} \in (\text{int dom } f)^\mathbb{N}$ , let  $C \subset \mathcal{X}$  be such that  $C \cap \text{int dom } f \neq \emptyset$ , and let  $x \in C \cap \text{int dom } f$ . Suppose that  $(x_n)_{n \in \mathbb{N}}$  is quasi-Bregman monotone with respect to  $C$  relative to  $(f_n)_{n \in \mathbb{N}}$ . Then the following hold.

(i)  $(D^{f_n}(x, x_n))_{n \in \mathbb{N}}$  converges.

(ii) Suppose that  $D^f(x, \cdot)$  is coercive. Then  $(x_n)_{n \in \mathbb{N}}$  is bounded.

**Proposition 2.5** [25] Let  $\mathcal{X}$  be a reflexive real Banach space, let  $f \in \Gamma_0(\mathcal{X})$  be Gâteaux differentiable on  $\text{int dom } f \neq \emptyset$ , let  $(x_n)_{n \in \mathbb{N}} \in (\text{int dom } f)^\mathbb{N}$ , let  $C \subset \mathcal{X}$  be such that  $C \cap \text{int dom } f \neq \emptyset$ , let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , let  $\alpha \in ]0, +\infty[$ , and let  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{P}_\alpha(f)$  be such that  $(\forall n \in \mathbb{N}) (1 + \eta_n)f_n \succcurlyeq f_{n+1}$ . Suppose that  $(x_n)_{n \in \mathbb{N}}$  is quasi-Bregman monotone with respect to  $C$  relative to  $(f_n)_{n \in \mathbb{N}}$ , that there exists  $g \in \mathcal{F}(f)$  such that for every  $n \in \mathbb{N}$ ,  $g \succcurlyeq f_n$ , and that, for every  $y_1 \in \mathcal{X}$  and every  $y_2 \in \mathcal{X}$ ,

$$\begin{cases} y_1 \in \mathfrak{W}(x_n)_{n \in \mathbb{N}} \cap C \\ y_2 \in \mathfrak{W}(x_n)_{n \in \mathbb{N}} \cap C \\ (\langle y_1 - y_2, \nabla f_n(x_n) \rangle)_{n \in \mathbb{N}} \text{ converges} \end{cases} \implies y_1 = y_2. \quad (2.7)$$

Moreover, suppose that  $(\forall x \in \text{int dom } f) D^f(x, \cdot)$  is coercive. Then  $(x_n)_{n \in \mathbb{N}}$  converges weakly to a point in  $C \cap \text{int dom } f$  if and only if  $\mathfrak{W}(x_n)_{n \in \mathbb{N}} \subset C \cap \text{int dom } f$ .

**Proposition 2.6** [25] Let  $\mathcal{X}$  be a reflexive real Banach space, let  $f \in \Gamma_0(\mathcal{X})$  be a Legendre function, let  $\alpha \in ]0, +\infty[$ , let  $(f_n)_{n \in \mathbb{N}}$  be in  $\mathcal{P}_\alpha(f)$ , let  $(x_n)_{n \in \mathbb{N}} \in (\text{int dom } f)^\mathbb{N}$ , and let  $C$  be a closed convex subset of  $\mathcal{X}$  such that  $C \cap \text{int dom } f \neq \emptyset$ . Suppose that  $(x_n)_{n \in \mathbb{N}}$  is stationarily quasi-Bregman monotone with respect to  $C$  relative to  $(f_n)_{n \in \mathbb{N}}$ , that  $f$  satisfies Condition 2.3, and that  $(\forall x \in \text{int dom } f) D^f(x, \cdot)$  is coercive. In addition, suppose that there exists  $\beta \in ]0, +\infty[$  such that  $(\forall n \in \mathbb{N}) \beta f \succcurlyeq f_n$ . Then  $(x_n)_{n \in \mathbb{N}}$  converges strongly to a point in  $C \cap \overline{\text{dom } f}$  if and only if  $\lim D_C^f(x_n) = 0$ .

We discuss some basic properties of a type of Bregman distance-based proximity operators in the following proposition.

**Proposition 2.7** Let  $\mathcal{X}$  be a reflexive real Banach space, let  $f \in \Gamma_0(\mathcal{X})$  be Gâteaux differentiable on  $\text{int dom } f \neq \emptyset$ , let  $\varphi \in \Gamma_0(\mathcal{X})$ , and let

$$\begin{aligned} \text{Prox}_\varphi^f : \mathcal{X}^* &\rightarrow 2^\mathcal{X} \\ x^* &\mapsto \{x \in \mathcal{X} \mid \varphi(x) + f(x) - \langle x, x^* \rangle = \min (\varphi + f - x^*)(\mathcal{X}) < +\infty\} \end{aligned} \quad (2.8)$$

be  $f$ -proximity operator of  $\varphi$ . Then the following hold.

- (i)  $\text{ran Prox}_\varphi^f \subset \text{dom } f \cap \text{dom } \varphi$  and  $\text{Prox}_\varphi^f = (\partial(f + \varphi))^{-1}$ .
- (ii) Suppose that  $\text{dom } \varphi \cap \text{int dom } f \neq \emptyset$  and that  $\text{dom } \partial f \cap \text{dom } \partial \varphi \subset \text{int dom } f$ . Then the following hold.
  - (a)  $\text{ran Prox}_\varphi^f \subset \text{int dom } f$  and  $\text{Prox}_\varphi^f = (\nabla f + \partial \varphi)^{-1}$ .
  - (b)  $\text{int}(\text{dom } f^* + \text{dom } \varphi^*) \subset \text{dom Prox}_\varphi^f$ .
  - (c) Suppose that  $f|_{\text{int dom } f}$  is strictly convex. Then  $\text{Prox}_\varphi^f$  is single-valued on its domain.

*Proof.* Let us fix  $x^* \in \mathcal{X}^*$  and define  $f_{x^*} : \mathcal{X} \rightarrow ]-\infty, +\infty] : x \mapsto f(x) - \langle x, x^* \rangle + f^*(x^*)$ . Then  $\text{dom } f_{x^*} = \text{dom } f$  and  $\varphi + f_{x^*} \in \Gamma_0(\mathcal{X})$ . Moreover,  $\partial(\varphi + f_{x^*}) = \partial(\varphi + f) - x^*$ .

(i): By definition,  $\text{ran Prox}_\varphi^f \subset \text{dom } f \cap \text{dom } \varphi$ . Suppose that  $\text{dom } f \cap \text{dom } \varphi \neq \emptyset$  and let  $x \in \text{dom } f \cap \text{dom } \varphi$ . Then

$$\begin{aligned} x \in \text{Prox}_\varphi^f x^* &\Leftrightarrow 0 \in \partial(\varphi + f_{x^*})(x) \\ &\Leftrightarrow 0 \in \partial(\varphi + f)(x) - x^* \\ &\Leftrightarrow x^* \in \partial(\varphi + f)(x) \\ &\Leftrightarrow x \in (\partial(\varphi + f))^{-1}(x^*). \end{aligned} \quad (2.9)$$

(ii): Suppose that  $x^* \in \text{int}(\text{dom } f^* + \text{dom } \varphi^*)$ . Since  $\text{dom } \varphi \cap \text{int dom } f \neq \emptyset$ , it follows from [1, Theorem 1.1] and [28, Theorem 2.1.3(ix)] that

$$x^* \in \text{int}(\text{dom } f^* + \text{dom } \varphi^*) = \text{int dom } (f + \varphi)^*. \quad (2.10)$$

(ii)(a): Since  $\text{dom } \varphi \cap \text{int dom } f \neq \emptyset$ ,  $\partial(\varphi + f) = \partial\varphi + \partial f$  by [1, Corollary 2.1], and hence (i) yields

$$\text{ran Prox}_\varphi^f = \text{dom } \partial(f + \varphi) = \text{dom } (\partial f + \partial\varphi) = \text{dom } \partial f \cap \text{dom } \partial\varphi \subset \text{int dom } f. \quad (2.11)$$

In turn,  $\text{ran Prox}_\varphi^f \subset \text{dom } \varphi \cap \text{int dom } f$ . We now prove that  $\text{Prox}_\varphi^f = (\nabla f + \partial\varphi)^{-1}$ . Note that  $\text{dom } (\nabla f + \partial\varphi) \subset \text{dom } \varphi \cap \text{int dom } f$ . Let  $x \in \text{dom } \varphi \cap \text{int dom } f$ . Then  $\partial(f + \varphi)(x) = \partial f(x) + \partial\varphi(x) = \nabla f(x) + \partial\varphi(x)$  and therefore,

$$x \in \text{Prox}_\varphi^f x^* \Leftrightarrow x^* \in \partial(f + \varphi)(x) = \nabla f(x) + \partial\varphi(x) \Leftrightarrow x \in (\nabla f + \partial\varphi)^{-1}(x^*). \quad (2.12)$$

(ii)(b): We derive from (2.10) and [5, Fact 3.1] that  $\varphi + f_{x^*}$  is coercive. Hence, by [28, Theorem 2.5.1],  $\varphi + f_{x^*}$  admits at least one minimizer, i.e.,  $x^* \in \text{dom Prox}_\varphi^f$ .

(ii)(c): Since  $f|_{\text{int dom } f}$  is strictly convex, so is  $(\varphi + f_{x^*})|_{\text{int dom } f}$  and thus, in view of (ii)(b),  $\varphi + f_{x^*}$  admits a unique minimizer on  $\text{int dom } f$ . However, since

$$\text{Argmin}(\varphi + f_{x^*}) = \text{ran Prox}_\varphi^f \subset \text{int dom } f, \quad (2.13)$$

it follows that  $\varphi + f_{x^*}$  admits a unique minimizer and that  $\text{Prox}_\varphi^f$  is therefore single-valued.  $\square$

**Proposition 2.8** *Let  $m$  be a strictly positive integer, let  $(\mathcal{X}_i)_{1 \leq i \leq m}$  be reflexive real Banach spaces, and let  $\mathcal{X}$  be the vector product space  $\bigtimes_{i=1}^m \mathcal{X}_i$  equipped with the norm  $x = (x_i)_{1 \leq i \leq m} \mapsto \sqrt{\sum_{i=1}^m \|x_i\|^2}$ . For every  $i \in \{1, \dots, m\}$ , let  $f_i \in \Gamma_0(\mathcal{X}_i)$  be a Legendre function and let  $\varphi_i \in \Gamma_0(\mathcal{X}_i)$  be such that  $\text{dom } \varphi_i \cap \text{int dom } f_i \neq \emptyset$ . Set  $f : \mathcal{X} \rightarrow ]-\infty, +\infty] : x \mapsto \sum_{i=1}^m f_i(x_i)$  and  $\varphi : \mathcal{X} \rightarrow ]-\infty, +\infty] : x \mapsto \sum_{i=1}^m \varphi_i(x_i)$ . Then*

$$\left( \forall x^* = (x_i^*)_{1 \leq i \leq m} \in \bigtimes_{i=1}^m \text{int}(\text{dom } f_i^* + \text{dom } \varphi_i^*) \right) \quad \text{Prox}_\varphi^f x^* = (\text{Prox}_{\varphi_i}^{f_i} x_i^*)_{1 \leq i \leq m}. \quad (2.14)$$

*Proof.* First, we observe that  $\mathcal{X}^*$  is the vector product space  $\bigtimes_{i=1}^m \mathcal{X}_i^*$  equipped with the norm  $x^* = (x_i^*)_{1 \leq i \leq m} \mapsto \sqrt{\sum_{i=1}^m \|x_i^*\|^2}$ . Next, we derive from the definition of  $f$  that  $\text{dom } f = \bigtimes_{i=1}^m \text{dom } f_i$  and that

$$\partial f : \mathcal{X} \rightarrow 2^{\mathcal{X}^*} : (x_i)_{1 \leq i \leq m} \mapsto \bigtimes_{i=1}^m \partial f_i(x_i). \quad (2.15)$$

Thus,  $\partial f$  is single-valued on

$$\text{dom } \partial f = \bigtimes_{i=1}^m \text{dom } \partial f_i = \bigtimes_{i=1}^m \text{int dom } f_i = \text{int} \left( \bigtimes_{i=1}^m \text{dom } f_i \right) = \text{int dom } f. \quad (2.16)$$

Likewise, since

$$f^* : \mathcal{X}^* \rightarrow ]-\infty, +\infty] : (x_i^*)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m f_i^*(x_i^*), \quad (2.17)$$

we deduce that  $\partial f^*$  is single-valued on  $\text{dom } \partial f^* = \text{int dom } f^*$ . Consequently, [5, Theorems 5.4 and 5.6] assert that

$$f \text{ is a Legendre function.} \quad (2.18)$$

In addition,

$$\text{dom } \varphi \cap \text{int dom } f = \left( \bigtimes_{i=1}^m \text{dom } \varphi_i \right) \cap \left( \bigtimes_{i=1}^m \text{int dom } f_i \right) = \bigtimes_{i=1}^m (\text{dom } \varphi_i \cap \text{int dom } f_i) \neq \emptyset. \quad (2.19)$$

Hence, Proposition 2.7(ii)(b)&(ii)(c) assert that  $\text{int}(\text{dom } f^* + \text{dom } \varphi^*) \subset \text{dom } \text{Prox}_{\varphi}^f$  and  $\text{Prox}_{\varphi}^f$  is single-valued on its domain. Now set  $x = \text{Prox}_{\varphi}^f x^*$  and  $q = (\text{Prox}_{\varphi_i}^{f_i} x_i^*)_{1 \leq i \leq m}$ . We derive from Proposition 2.7(ii)(a) that

$$x = \text{Prox}_{\varphi}^f x^* \Leftrightarrow x = (\nabla f + \partial \varphi)^{-1}(x^*) \Leftrightarrow x^* - \nabla f(x) \in \partial \varphi(x). \quad (2.20)$$

Consequently, by invoking (1.4), we get

$$(\forall z \in \text{dom } \varphi) \quad \langle z - x, x^* - \nabla f(x) \rangle + \varphi(x) \leq \varphi(z). \quad (2.21)$$

Upon setting  $z = q$  in (2.21), we obtain

$$\langle q - x, x^* - \nabla f(x) \rangle + \varphi(x) \leq \varphi(q). \quad (2.22)$$

For every  $i \in \{1, \dots, m\}$ , let us set  $q_i = \text{Prox}_{\varphi_i}^{f_i} x_i^*$ . The same characterization as in (2.21) yields

$$(\forall i \in \{1, \dots, m\})(\forall z_i \in \text{dom } \varphi_i) \quad \langle z_i - q_i, x_i^* - \nabla f_i(q_i) \rangle + \varphi_i(q_i) \leq \varphi_i(z_i). \quad (2.23)$$

By summing these inequalities over  $i \in \{1, \dots, m\}$ , we obtain

$$(\forall z \in \text{dom } \varphi) \quad \langle z - q, x^* - \nabla f(q) \rangle + \varphi(q) \leq \varphi(z). \quad (2.24)$$

Upon setting  $z = x$  in (2.24), we get

$$\langle x - q, \nabla f(x) - \nabla f(q) \rangle + \varphi(q) \leq \varphi(x). \quad (2.25)$$

Adding (2.22) and (2.25) yields

$$\langle x - q, \nabla f(x) - \nabla f(q) \rangle \leq 0. \quad (2.26)$$

Now suppose that  $x \neq q$ . Since  $f|_{\text{int dom } f}$  is strictly convex, it follows from [28, Theorem 2.4.4(ii)] that  $\nabla f$  is strictly monotone, i.e.,

$$\langle x - q, \nabla f(x) - \nabla f(q) \rangle > 0, \quad (2.27)$$

and we reach a contradiction.  $\square$

In Hilbert spaces, the operator defined in (2.8) reduces to the Moreau's usual proximity operator  $\text{prox}_{\varphi}$  [24] if  $f = \|\cdot\|^2/2$ . We provide illustrations of instances in the standard Euclidean space  $\mathbb{R}^m$  in which  $\text{Prox}_{\varphi}^f$  is easier to evaluate than  $\text{prox}_{\varphi}$ .

**Example 2.9** Let  $\gamma \in ]0, +\infty[$ , let  $\phi \in \Gamma_0(\mathbb{R})$  be such that  $\text{dom } \phi \cap ]0, +\infty[ \neq \emptyset$ , and let  $\vartheta$  be Boltzmann-Shannon entropy, i.e.,

$$\vartheta: \xi \mapsto \begin{cases} \xi \ln \xi - \xi, & \text{if } \xi \in ]0, +\infty[; \\ 0, & \text{if } \xi = 0; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.28)$$

Set  $\varphi: (\xi_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m \phi(\xi_i)$  and  $f: (\xi_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m \vartheta(\xi_i)$ . Note that  $f$  is a supercoercive Legendre function [4, Sections 5 and 6], and hence, Proposition 2.7(ii)(b) asserts that  $\text{dom } \text{Prox}_{\varphi}^f = \mathbb{R}^m$ . Let  $(\xi_i)_{1 \leq i \leq m} \in \mathbb{R}^m$ , set  $(\eta_i)_{1 \leq i \leq m} = \text{Prox}_{\gamma\varphi}^f(\xi_i)_{1 \leq i \leq m}$ , let  $W$  be the Lambert function [19], i.e., the inverse of  $\xi \mapsto \xi e^{\xi}$  on  $[0, +\infty[$ , and let  $i \in \{1, \dots, m\}$ . Then  $\eta_i$  can be computed as follows.

(i) Let  $\omega \in \mathbb{R}$  and suppose that

$$\phi: \xi \mapsto \begin{cases} \xi \ln \xi - \omega \xi, & \text{if } \xi \in ]0, +\infty[; \\ 0, & \text{if } \xi = 0; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.29)$$

Then  $\eta_i = e^{(\xi_i + \omega - 1)/(\gamma + 1)}$ .

(ii) Let  $p \in [1, +\infty[$  and suppose that either  $\phi = |\cdot|^p/p$  or

$$\phi: \xi \mapsto \begin{cases} \xi^p/p, & \text{if } \xi \in [0, +\infty[; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.30)$$

Then

$$\eta_i = \begin{cases} \left( \frac{W(\gamma(p-1)e^{(p-1)\xi_i})}{\gamma(p-1)} \right)^{\frac{1}{p-1}}, & \text{if } p \in ]1, +\infty[; \\ e^{\xi_i - \gamma}, & \text{if } p = 1. \end{cases} \quad (2.31)$$

(iii) Let  $p \in [1, +\infty[$  and suppose that

$$\phi: \xi \mapsto \begin{cases} \xi^{-p}/p, & \text{if } \xi \in ]0, +\infty[; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.32)$$

Then

$$\eta_i = \left( \frac{W(\gamma(p+1)e^{-(p+1)\xi_i})}{\gamma(p+1)} \right)^{\frac{-1}{p+1}}. \quad (2.33)$$



(iv) Let  $p \in ]0, 1[$  and suppose that

$$\phi : \xi \mapsto \begin{cases} -\xi^p/p, & \text{if } \xi \in [0, +\infty[; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.34)$$

Then

$$\eta_i = \left( \frac{W(\gamma(1-p)e^{(p-1)\xi_i})}{\gamma(1-p)} \right)^{\frac{1}{p-1}}. \quad (2.35)$$

**Example 2.10** Let  $\phi \in \Gamma_0(\mathbb{R})$  be such that  $\text{dom } \phi \cap ]0, 1[ \neq \emptyset$  and let  $\vartheta$  be Fermi-Dirac entropy, i.e.,

$$\vartheta : \xi \mapsto \begin{cases} \xi \ln \xi - (1 - \xi) \ln(1 - \xi), & \text{if } \xi \in ]0, 1[; \\ 0 & \text{if } \xi \in \{0, 1\}; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.36)$$

Set  $\varphi : (\xi_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m \phi(\xi_i)$  and  $f : (\xi_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m \vartheta(\xi_i)$ . Note that  $f$  is a cofinite Legendre function [4, Sections 5 and 6], and hence Proposition 2.7(ii)(b) asserts that  $\text{dom Prox}_{\varphi}^f = \mathbb{R}^m$ . Let  $(\xi_i)_{1 \leq i \leq m} \in \mathbb{R}^m$ , set  $(\eta_i)_{1 \leq i \leq m} = \text{Prox}_{\varphi}^f(\xi_i)_{1 \leq i \leq m}$ , and let  $i \in \{1, \dots, m\}$ . Then  $\eta_i$  can be computed as follows.

(i) Let  $\omega \in \mathbb{R}$  and suppose that

$$\phi : \xi \mapsto \begin{cases} \xi \ln \xi - \omega \xi, & \text{if } \xi \in ]0, +\infty[; \\ 0, & \text{if } \xi = 0; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.37)$$

$$\text{Then } \eta_i = -e^{\xi_i + \omega - 1} / 2 + \sqrt{e^{2(\xi_i + \omega - 1)} / 4 + e^{\xi_i + \omega - 1}}.$$

(ii) Suppose that

$$\phi : \xi \mapsto \begin{cases} (1 - \xi) \ln(1 - \xi) + \xi, & \text{if } \xi \in ]-\infty, 1[; \\ 1, & \text{if } \xi = 1; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.38)$$

$$\text{Then } \eta_i = 1 + e^{-\xi_i} / 2 - \sqrt{e^{-\xi_i} + e^{-2\xi_i} / 4}.$$

**Example 2.11** Let  $f : (\xi_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m \vartheta(\xi_i)$ , where  $\vartheta$  is Hellinger-like function, i.e.,

$$\vartheta : \xi \mapsto \begin{cases} -\sqrt{1 - \xi^2}, & \text{if } \xi \in [-1, 1]; \\ +\infty, & \text{otherwise,} \end{cases} \quad (2.39)$$

let  $\gamma \in ]0, +\infty[$ , and let  $\varphi = f$ . Since  $f$  is a cofinite Legendre function [4, Sections 5 and 6], Proposition 2.7(ii)(b) asserts that  $\text{dom Prox}_{\gamma\varphi}^f = \mathbb{R}^m$ . Let  $(\xi_i)_{1 \leq i \leq m} \in \mathbb{R}^m$ , and set  $(\eta_i)_{1 \leq i \leq m} = \text{Prox}_{\gamma\varphi}^f(\xi_i)_{1 \leq i \leq m}$ . Then  $(\forall i \in \{1, \dots, m\}) \eta_i = \xi_i / \sqrt{(\gamma + 1)^2 + \xi_i^2}$ .

**Example 2.12** Let  $\gamma \in ]0, +\infty[$ , let  $\phi \in \Gamma_0(\mathbb{R})$  be such that  $\text{dom } \phi \cap ]0, +\infty[ \neq \emptyset$ , and let  $\vartheta$  be Burg entropy, i.e.,

$$\vartheta: \xi \mapsto \begin{cases} -\ln \xi, & \text{if } \xi \in ]0, +\infty[; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.40)$$

Set  $\varphi: (\xi_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m \phi(\xi_i)$  and  $f: (\xi_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m \vartheta(\xi_i)$ , let  $(\xi_i)_{1 \leq i \leq m} \in \mathbb{R}^m$ , and set  $(\eta_i)_{1 \leq i \leq m} = \text{Prox}_{\varphi}^f(\xi_i)_{1 \leq i \leq m}$ . Let  $i \in \{1, \dots, m\}$ . Then  $\eta_i$  can be computed as follows.

- (i) Suppose that  $\phi = \vartheta$  and  $\xi_i \in ]-\infty, 0]$ . Then  $\eta_i = -(1 + \gamma)^{-1} \xi_i$ .
- (ii) Suppose that  $\phi: \xi \mapsto \alpha|\xi|$  and  $\xi_i \in ]-\infty, \gamma\alpha]$ . Then  $\eta_i = (\gamma\alpha - \xi_i)^{-1}$ .

The following result will be used subsequently.

**Lemma 2.13** Let  $\mathcal{X}$  be a reflexive real Banach space, let  $f \in \Gamma_0(\mathcal{X})$  be a Legendre function, let  $x \in \text{int dom } f$ , and let  $(x_n)_{n \in \mathbb{N}} \in (\text{int dom } f)^{\mathbb{N}}$ . Suppose that  $(D^f(x, x_n))_{n \in \mathbb{N}}$  is bounded, that  $\text{dom } f^*$  is open, and that  $\nabla f^*$  is weakly sequentially continuous. Then  $\mathfrak{W}(x_n)_{n \in \mathbb{N}} \subset \text{int dom } f$ .

*Proof.* [25, Proof of Theorem 4.1].  $\square$

### 3 Forward-backward splitting in Banach spaces

The first result in this section is a version of the forward-backward splitting algorithm in reflexive real Banach spaces which employs different Bregman distance-based proximity operators over the iterations.

**Theorem 3.1** Consider the setting of Problem 1.1 and let  $f \in \Gamma_0(\mathcal{X})$  be a Legendre function such that  $\mathcal{S} \cap \text{int dom } f \neq \emptyset$ ,  $L(\text{int dom } f) \subset \text{int dom } \psi$ , and  $f \succcurlyeq \beta\psi \circ L$  for some  $\beta \in ]0, +\infty[$ . Let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , let  $\alpha \in ]0, +\infty[$ , and let  $(f_n)_{n \in \mathbb{N}}$  be Legendre functions in  $\mathcal{P}_\alpha(f)$  such that

$$(\forall n \in \mathbb{N}) \quad (1 + \eta_n)f_n \succcurlyeq f_{n+1}. \quad (3.1)$$

Suppose that either  $-L^*(\text{ran } \nabla \psi) \subset \text{dom } \varphi^*$  or  $(\forall n \in \mathbb{N}) f_n$  is cofinite. Let  $\varepsilon \in ]0, \alpha\beta/(\alpha\beta + 1)[$  and let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$  such that

$$(\forall n \in \mathbb{N}) \quad \varepsilon \leq \gamma_n \leq \alpha\beta(1 - \varepsilon) \quad \text{and} \quad (1 + \eta_n)\gamma_n - \gamma_{n+1} \leq \alpha\beta\eta_n. \quad (3.2)$$

Furthermore, let  $x_0 \in \text{int dom } f$  and iterate

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \text{Prox}_{\gamma_n \varphi}^{f_n}(\nabla f_n(x_n) - \gamma_n L^* \nabla \psi(Lx_n)). \quad (3.3)$$

Suppose in addition that  $(\forall x \in \text{int dom } f) D^f(x, \cdot)$  is coercive. Then  $(x_n)_{n \in \mathbb{N}}$  is a bounded sequence in  $\text{int dom } f$  and  $\mathfrak{W}(x_n)_{n \in \mathbb{N}} \subset \mathcal{S}$ . Moreover, there exists  $\bar{x} \in \mathcal{S}$  such that the following hold.

- (i) Suppose that  $\mathcal{S} \cap \overline{\text{dom} f}$  is a singleton. Then  $x_n \rightarrow \bar{x}$ .
- (ii) Suppose that there exists  $g \in \mathcal{F}(f)$  such that for every  $n \in \mathbb{N}$ ,  $g \succ f_n$ , and that, for every  $y_1 \in \mathcal{X}$  and every  $y_2 \in \mathcal{X}$ ,

$$\begin{cases} y_1 \in \mathcal{W}(x_n)_{n \in \mathbb{N}} \\ y_2 \in \mathcal{W}(x_n)_{n \in \mathbb{N}} \\ (\langle y_1 - y_2, \nabla f_n(x_n) - \gamma_n L^* \nabla \psi(Lx_n) \rangle)_{n \in \mathbb{N}} \text{ converges} \end{cases} \Rightarrow y_1 = y_2. \quad (3.4)$$

In addition, suppose that one of the following holds.

- (a)  $\mathcal{S} \subset \text{int dom} f$ .
- (b)  $\text{dom} f^*$  is open and  $\nabla f^*$  is weakly sequentially continuous.

Then  $x_n \rightarrow \bar{x}$ .

- (iii) Suppose that  $f$  satisfies Condition 2.3 and that one of the following holds.

- (a)  $\varphi$  is uniformly convex at  $\bar{x}$ .
- (b)  $\psi$  is uniformly convex at  $L\bar{x}$  and there exists  $\kappa \in ]0, +\infty[$  such that  $(\forall x \in \mathcal{X}) \|Lx\| \geq \kappa \|x\|$ .
- (c)  $\lim D_s^f(x_n) = 0$  and there exists  $\mu \in ]0, +\infty[$  such that  $(\forall n \in \mathbb{N}) \mu f \succ f_n$ .

Then  $x_n \rightarrow \bar{x}$ .

*Proof.* We first derive from Proposition 2.7(ii)(c) that the operators  $(\text{Prox}_{\gamma_n \varphi}^f)_{n \in \mathbb{N}}$  are single-valued on their domains. We also note that  $x_0 \in \text{int dom} f$ . Suppose that  $x_n \in \text{int dom} f$  for some  $n \in \mathbb{N}$ . If  $f_n$  is cofinite then Proposition 2.7(ii)(b) yields

$$\nabla f_n(x_n) - \gamma_n L^* \nabla \psi(Lx_n) \in \mathcal{X}^* = \text{dom Prox}_{\gamma_n \varphi}^{f_n}. \quad (3.5)$$

Otherwise,

$$\begin{aligned} \nabla f_n(x_n) - \gamma_n L^* \nabla \psi(Lx_n) &\in \text{int dom} f_n^* + \gamma_n \text{dom } \varphi^* = \text{int}(\text{int dom} f_n^* + \gamma_n \text{dom } \varphi^*) \\ &\subset \text{int}(\text{dom} f_n^* + \gamma_n \text{dom } \varphi^*) = \text{int}(\text{dom} f_n^* + \text{dom}(\gamma_n \varphi^*)). \end{aligned} \quad (3.6)$$

Since  $\text{int}(\text{dom} f_n^* + \text{dom}(\gamma_n \varphi^*)) \subset \text{dom Prox}_{\gamma_n \varphi}^f$  by Proposition 2.7(ii)(b), we deduce from (3.3), (3.5), (3.6), and Proposition 2.7(ii)(a) that  $x_{n+1}$  is a well-defined element in  $\text{ran Prox}_{\gamma \varphi}^{f_n} = \text{dom } \partial \varphi \cap \text{int dom} f_n = \text{dom } \partial \varphi \cap \text{int dom} f \subset \text{int dom} f$ . By reasoning by induction, we conclude that

$$(x_n)_{n \in \mathbb{N}} \in (\text{int dom} f)^{\mathbb{N}} \text{ is well-defined.} \quad (3.7)$$

Next, let us set  $\Phi = \varphi + \psi \circ L$  and

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad g_n: \mathcal{X} &\rightarrow ]-\infty, +\infty] \\ x &\mapsto \begin{cases} f_n(x) - \gamma_n \psi(Lx), & \text{if } x \in \text{int dom} f; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \quad (3.8)$$

Since  $L(\text{int dom } f) \subset \text{int dom } \psi$ , it follows from (3.8) that  $(\forall n \in \mathbb{N})$   $g_n$  is Gâteaux differentiable on  $\text{dom } g_n = \text{int dom } g_n = \text{int dom } f$ . Since  $\psi$  is continuous on  $\text{int dom } \psi \supset L(\text{int dom } f)$  and the functions  $(f_n)_{n \in \mathbb{N}}$  are continuous on  $\text{int dom } f$  [26, Proposition 3.3], we deduce that  $(\forall n \in \mathbb{N})$   $g_n$  is continuous on  $\text{dom } g_n$ . In addition,

$$(\forall n \in \mathbb{N}) \quad g_n - \varepsilon \alpha f = (1 - \varepsilon)(f_n - \alpha \beta \psi \circ L) + \varepsilon(f_n - \alpha f) + (\alpha \beta(1 - \varepsilon) - \gamma_n) \psi \circ L. \quad (3.9)$$

Note that  $f \succcurlyeq \beta \psi \circ L$  and  $(\forall n \in \mathbb{N})$   $f_n \succcurlyeq \alpha f$ . Hence, (3.9) yields

$$(\forall n \in \mathbb{N}) \quad f_n \succcurlyeq \alpha \beta \psi \circ L, \quad (3.10)$$

and hence, we deduce from (3.2) and (3.9) that  $(\forall n \in \mathbb{N})$   $g_n \succcurlyeq \varepsilon \alpha f$ . In turn,

$$\begin{aligned} (\forall n \in \mathbb{N})(\forall x \in \text{dom } g_n)(\forall y \in \text{dom } g_n) \quad & \langle x - y, \nabla g_n(x) - \nabla g_n(y) \rangle \\ & = D^{g_n}(x, y) + D^{g_n}(y, x) \geq \varepsilon \alpha (D^f(x, y) + D^f(y, x)) \geq 0, \end{aligned} \quad (3.11)$$

and it therefore follows from [28, Theorem 2.1.11] that  $(\forall n \in \mathbb{N})$   $g_n$  is convex. Consequently,

$$(\forall n \in \mathbb{N}) \quad g_n \in \mathcal{P}_{\varepsilon \alpha}(f). \quad (3.12)$$

Set  $\omega = 1 + 1/\varepsilon$ . Then

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad (1 + \omega \eta_n)g_n - g_{n+1} &= (1 + \omega \eta_n)(f_n - \gamma_n \psi \circ L) - (f_{n+1} - \gamma_{n+1} \psi \circ L) \\ &= (1 + \eta_n)f_n - f_{n+1} + \eta_n \varepsilon^{-1} (f_n - (\gamma_n + \varepsilon \alpha \beta) \psi \circ L) \\ &\quad + (\alpha \beta \eta_n + \gamma_{n+1} - (1 + \eta_n)\gamma_n) \psi \circ L. \end{aligned} \quad (3.13)$$

We thus derive from (3.2) and (3.10) that

$$(\forall n \in \mathbb{N}) \quad (1 + \omega \eta_n)g_n \succcurlyeq g_{n+1}. \quad (3.14)$$

By invoking (3.3) and Proposition 2.7(ii)(a), we get

$$(\forall n \in \mathbb{N}) \quad \nabla f_n(x_n) - \gamma_n L^* \nabla \psi(Lx_n) \in \nabla f_n(x_{n+1}) + \gamma_n \partial \varphi(x_{n+1}), \quad (3.15)$$

and therefore,

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad \nabla f_n(x_n) - \gamma_n L^* \nabla \psi(Lx_n) &\in \nabla f_n(x_{n+1}) - \gamma_n L^* \nabla \psi(Lx_{n+1}) \\ &\quad + \gamma_n (\partial \varphi(x_{n+1}) + L^* \nabla \psi(Lx_{n+1})). \end{aligned} \quad (3.16)$$

Since [28, Theorem 2.4.2(vii)–(viii)] yield

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad \partial \varphi(x_{n+1}) + L^* \nabla \psi(Lx_{n+1}) &\subset \partial \varphi(x_{n+1}) + L^* (\partial \psi(Lx_{n+1})) \\ &\subset \partial (\varphi + \psi \circ L)(x_{n+1}) = \partial \Phi(x_{n+1}), \end{aligned} \quad (3.17)$$

we deduce from (3.16) that

$$(\forall n \in \mathbb{N}) \quad \nabla g_n(x_n) - \nabla g_n(x_{n+1}) \in \gamma_n \partial \Phi(x_{n+1}). \quad (3.18)$$

By appealing to (1.4) and (3.18), we get

$$(\forall x \in \text{dom } \Phi \cap \text{dom } f)(\forall n \in \mathbb{N}) \quad \gamma_n^{-1} \langle x - x_{n+1}, \nabla g_n(x_n) - \nabla g_n(x_{n+1}) \rangle + \Phi(x_{n+1}) \leq \Phi(x), \quad (3.19)$$

and hence, by [6, Proposition 2.3(ii)],

$$(\forall x \in \text{dom } \Phi \cap \text{dom } f)(\forall n \in \mathbb{N}) \quad \gamma_n^{-1} (D^{g_n}(x, x_{n+1}) + D^{g_n}(x_{n+1}, x_n) - D^{g_n}(x, x_n)) + \Phi(x_{n+1}) \leq \Phi(x). \quad (3.20)$$

In particular,

$$(\forall x \in \mathcal{S} \cap \text{dom } f)(\forall n \in \mathbb{N}) \quad D^{g_n}(x, x_{n+1}) + D^{g_n}(x_{n+1}, x_n) - D^{g_n}(x, x_n) \leq 0. \quad (3.21)$$

By using (3.14), we deduce from (3.21) that

$$(\forall x \in \mathcal{S} \cap \text{dom } f)(\forall n \in \mathbb{N}) \quad D^{g_{n+1}}(x, x_{n+1}) + (1 + \omega \eta_n) D^{g_n}(x_{n+1}, x_n) \leq (1 + \omega \eta_n) D^{g_n}(x, x_n), \quad (3.22)$$

and therefore,

$$(\forall x \in \mathcal{S} \cap \text{dom } f)(\forall n \in \mathbb{N}) \quad D^{g_{n+1}}(x, x_{n+1}) \leq (1 + \omega \eta_n) D^{g_n}(x, x_n). \quad (3.23)$$

This shows that  $(x_n)_{n \in \mathbb{N}}$  is stationarily quasi-Bregman monotone with respect to  $\mathcal{S}$  relative to  $(g_n)_{n \in \mathbb{N}}$ . Hence, we deduce from Proposition 2.4(ii) that

$$(x_n)_{n \in \mathbb{N}} \in (\text{int dom } f)^{\mathbb{N}} \quad \text{is bounded} \quad (3.24)$$

and, since  $\mathcal{X}$  is reflexive,

$$\mathfrak{W}(x_n)_{n \in \mathbb{N}} \neq \emptyset. \quad (3.25)$$

In addition, we derive from (3.23) and Proposition 2.4(i) that

$$(\forall x \in \mathcal{S} \cap \text{int dom } f) \quad (D^{g_n}(x, x_n))_{n \in \mathbb{N}} \quad \text{converges}, \quad (3.26)$$

and thus, since (3.22) yields

$$\begin{aligned} (\forall x \in \mathcal{S} \cap \text{int dom } f)(\forall n \in \mathbb{N}) \quad 0 &\leq D^{g_n}(x_{n+1}, x_n) \\ &\leq (1 + \omega \eta_n) D^{f_n}(x_{n+1}, x_n) \\ &\leq (1 + \omega \eta_n) D^{f_n}(x, x_n) - D^{f_{n+1}}(x, x_{n+1}), \end{aligned} \quad (3.27)$$

we obtain

$$D^{g_n}(x_{n+1}, x_n) \rightarrow 0. \quad (3.28)$$

On the other hand, it follows from (3.12) that

$$(\forall n \in \mathbb{N}) \quad \varepsilon \alpha D^f(x_{n+1}, x_n) \leq D^{g_n}(x_{n+1}, x_n), \quad (3.29)$$

and hence, (3.28) yields

$$D^f(x_{n+1}, x_n) \rightarrow 0. \quad (3.30)$$

Now, it follows from (3.20) that

$$(\forall n \in \mathbb{N}) \quad \Phi(x_{n+1}) \leq \gamma_n^{-1} (D^{g_n}(x_n, x_{n+1}) + D^{g_n}(x_{n+1}, x_n)) + \Phi(x_{n+1}) \leq \Phi(x_n), \quad (3.31)$$

which shows that  $(\Phi(x_n))_{n \in \mathbb{N}}$  is decreasing and hence, since it is bounded from below by  $\inf \Phi(\mathcal{X})$ , it is convergent. However, (3.20) and (3.23) yield

$$\begin{aligned} & (\forall x \in \mathcal{S} \cap \text{int dom } f)(\forall n \in \mathbb{N}) \\ & \varepsilon^{-1} \left( \frac{1}{1 + \omega \eta_n} D^{g_{n+1}}(x, x_{n+1}) + D^{g_n}(x_{n+1}, x_n) - D^{g_n}(x, x_n) \right) + \Phi(x_{n+1}) \\ & \leq \gamma_n^{-1} \left( \frac{1}{1 + \omega \eta_n} D^{g_{n+1}}(x, x_{n+1}) + D^{g_n}(x_{n+1}, x_n) - D^{g_n}(x, x_n) \right) + \Phi(x_{n+1}) \\ & \leq \Phi(x). \end{aligned} \quad (3.32)$$

Since  $\eta_n \rightarrow 0$ , by taking the limit in (3.32) and then using (3.26) and (3.28), we get

$$\inf \Phi(\mathcal{X}) \leq \lim \Phi(x_n) \leq \inf \Phi(\mathcal{X}), \quad (3.33)$$

and thus,

$$\Phi(x_n) \rightarrow \inf \Phi(\mathcal{X}). \quad (3.34)$$

We now show that

$$\mathfrak{W}(x_n)_{n \in \mathbb{N}} \subset \mathcal{S}. \quad (3.35)$$

To this end, suppose that  $x \in \mathfrak{W}(x_n)_{n \in \mathbb{N}}$ , i.e.,  $x_{k_n} \rightarrow x$ . Since  $\Phi$  is weakly lower semicontinuous [28, Theorem 2.2.1], by (3.34),

$$\inf \Phi(\mathcal{X}) \leq \Phi(x) \leq \liminf \Phi(x_{k_n}) = \lim \Phi(x_n) = \inf \Phi(\mathcal{X}). \quad (3.36)$$

This yields  $\Phi(x) = \inf \Phi(\mathcal{X})$ , i.e.,  $x \in \text{Argmin } \Phi = \mathcal{S}$ .

(i): Let  $\bar{x} \in \mathfrak{W}(x_n)_{n \in \mathbb{N}}$ . Since (3.24) and (3.35) imply that  $\mathfrak{W}(x_n)_{n \in \mathbb{N}} \subset \mathcal{S} \cap \overline{\text{dom } f}$ , we obtain  $\mathfrak{W}(x_n)_{n \in \mathbb{N}} = \{\bar{x}\}$ , and in turn, (3.25) yields  $x_n \rightarrow \bar{x}$ .

(ii): In view of (3.35) and Proposition 2.5, it suffices to show that  $\mathfrak{W}(x_n)_{n \in \mathbb{N}} \subset \text{int dom } f$ .

(ii)(a): We have  $\mathfrak{W}(x_n)_{n \in \mathbb{N}} \subset \mathcal{S} \subset \text{int dom } f$ .

(ii)(b): This follows from Lemma 2.13.

(iii): Let  $\bar{x} \in \mathcal{S} \cap \text{int dom } f$ . Since  $f$  satisfies Condition 2.3, (3.30) yields

$$x_{n+1} - x_n \rightarrow 0. \quad (3.37)$$

Now set

$$(\forall n \in \mathbb{N}) \quad z_n = x_{n+1} \quad \text{and} \quad z_n^* = \gamma_n^{-1}(\nabla g_n(x_n) - \nabla g_n(z_n)). \quad (3.38)$$

Then (3.18) and (3.37) imply that

$$(\forall n \in \mathbb{N}) \quad z_n^* \in \partial \Phi(z_n) \quad \text{and} \quad z_n - x_n \rightarrow 0. \quad (3.39)$$

Since (3.22) yields

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad D^{g_{n+1}}(\bar{x}, x_{n+1}) &= D^{g_{n+1}}(\bar{x}, z_n) \\ &\leq (1 + \omega \eta_n) D^{g_n}(\bar{x}, z_n) \\ &= (1 + \omega \eta_n) D^{g_n}(\bar{x}, x_{n+1}) \\ &\leq (1 + \omega \eta_n) D^{g_n}(\bar{x}, x_n), \end{aligned} \quad (3.40)$$

we deduce that

$$(\forall n \in \mathbb{N}) \quad (1 + \omega \eta_n)^{-1} D^{g_{n+1}}(\bar{x}, x_{n+1}) \leq D^{g_n}(\bar{x}, z_n) \leq D^{g_n}(\bar{x}, x_n). \quad (3.41)$$

Altogether, (3.26) and (3.41) yield

$$D^{g_n}(\bar{x}, z_n) - D^{g_n}(\bar{x}, x_n) \rightarrow 0. \quad (3.42)$$

In (3.19), by setting  $x = \bar{x}$ , we get

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad 0 &\leq \gamma_n \langle z_n - \bar{x}, z_n^* \rangle \\ &= \langle z_n - \bar{x}, \nabla g_n(x_n) - \nabla g_n(z_n) \rangle \\ &= D^{g_n}(\bar{x}, x_n) - D^{g_n}(\bar{x}, z_n) - D^{g_n}(z_n, x_n) \\ &\leq D^{g_n}(\bar{x}, x_n) - D^{g_n}(\bar{x}, z_n). \end{aligned} \quad (3.43)$$

By taking to the limit in (3.43) and using (3.42), we get

$$\langle z_n - \bar{x}, z_n^* \rangle \rightarrow 0. \quad (3.44)$$

(iii)(a): In this case  $\mathcal{S} = \{\bar{x}\}$ . Since  $\varphi$  is uniformly convex at  $\bar{x}$ ,  $\Phi$  is likewise and hence, there exists an increasing function  $\phi: [0, +\infty[ \rightarrow [0, +\infty]$  that vanishes only at 0 such that

$$(\forall n \in \mathbb{N})(\forall \tau \in ]0, 1[) \quad \Phi(\tau \bar{x} + (1 - \tau)z_n) + \tau(1 - \tau)\phi(\|z_n - \bar{x}\|) \leq \tau \Phi(\bar{x}) + (1 - \tau)\Phi(z_n). \quad (3.45)$$

It therefore follows from [28, Page 201] that  $\partial \Phi$  is uniformly monotone at  $\bar{x}$  and its modulus of convexity is  $\phi$ , i.e.,

$$(\forall n \in \mathbb{N}) \quad \langle z_n - \bar{x}, z_n^* \rangle \geq \phi(\|z_n - \bar{x}\|) \geq 0. \quad (3.46)$$

Altogether, (3.44) and (3.46) yield  $\phi(\|z_n - \bar{x}\|) \rightarrow 0$ , and thus,  $z_n \rightarrow \bar{x}$ . In turn, (3.39) yields  $x_n \rightarrow \bar{x}$ .

(iii)(b): By the same argument as in (iii)(a),  $\mathcal{S} = \{\bar{x}\}$  and there exists an increasing function  $\phi: [0, +\infty[ \rightarrow [0, +\infty]$  that vanishes only at 0 such that

$$(\forall n \in \mathbb{N}) \quad \langle z_n - \bar{x}, \nabla \psi(Lz_n) - \nabla \psi(L\bar{x}) \rangle \geq \phi(\|Lz_n - L\bar{x}\|). \quad (3.47)$$

In turn, it follows from (3.17) and [28, Theorem 2.4.2(iv)] that

$$(\forall n \in \mathbb{N}) \quad \langle z_n - \bar{x}, z_n^* \rangle \geq \phi(\|Lz_n - L\bar{x}\|). \quad (3.48)$$

This yields  $\phi(\|Lz_n - L\bar{x}\|) \rightarrow 0$ , and hence,  $Lz_n \rightarrow L\bar{x}$ . Since

$$(\forall n \in \mathbb{N}) \quad \|Lz_n - L\bar{x}\| \geq \kappa \|z_n - \bar{x}\|, \quad (3.49)$$

we obtain  $z_n \rightarrow \bar{x}$  and in turn, (3.39) yields  $x_n \rightarrow \bar{x}$ .

(iii)(c): First, we observe that  $\mathcal{S}$  is closed and convex since  $\Phi \in \Gamma_0(\mathcal{X})$ . Next, for every  $n \in \mathbb{N}$ , since  $\mu f \succcurlyeq f_n$ , we derive from (3.8) that  $\mu f \succcurlyeq g_n$ . Finally, the strong convergence follows from Proposition 2.6.  $\square$

The following corollary of Theorem 3.1 appears to be the first version of the forward-backward algorithm outside of Hilbert spaces.

**Theorem 3.2** *Consider the setting of Problem 1.1 and let  $f \in \Gamma_0(\mathcal{X})$  be a Legendre function such that  $\mathcal{S} \cap \text{int dom } f \neq \emptyset$ ,  $L(\text{int dom } f) \subset \text{int dom } \psi$ , and  $f \succcurlyeq \beta \psi \circ L$  for some  $\beta \in ]0, +\infty[$ . Suppose that either  $f$  is cofinite or  $-L^*(\text{ran } \nabla \psi) \subset \text{dom } \varphi^*$ , and that  $(\forall x \in \text{int dom } f)$   $D^f(x, \cdot)$  is coercive. Let  $\varepsilon \in ]0, \beta/(\beta + 1)[$ , let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , and let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$  such that*

$$(\forall n \in \mathbb{N}) \quad \varepsilon \leq \gamma_n \leq \beta(1 - \varepsilon) \quad \text{and} \quad (1 + \eta_n)\gamma_n - \gamma_{n+1} \leq \beta\eta_n. \quad (3.50)$$

Furthermore, let  $x_0 \in \text{int dom } f$  and iterate

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \text{Prox}_{\gamma_n \varphi}^f(\nabla f(x_n) - \gamma_n L^* \nabla \psi(Lx_n)). \quad (3.51)$$

Then there exists  $\bar{x} \in \mathcal{S}$  such that the following hold.

(i) Suppose that one of the following holds.

- (a)  $\mathcal{S} \cap \overline{\text{dom } f}$  is a singleton.
- (b)  $\nabla f$  and  $\nabla \psi$  are weakly sequentially continuous and  $\mathcal{S} \subset \text{int dom } f$ .
- (c)  $\text{dom } f^*$  is open and  $\nabla f$ ,  $\nabla f^*$ , and  $\nabla \psi$  are weakly sequentially continuous.

Then  $x_n \rightarrow \bar{x}$ .

(ii) Suppose that  $f$  satisfies Condition 2.3 and that one of the following holds.

- (a)  $\varphi$  is uniformly convex at  $\bar{x}$ .
- (b)  $\psi$  is uniformly convex at  $L\bar{x}$  and there exists  $\kappa \in ]0, +\infty[$  such that  $(\forall x \in \mathcal{X}) \quad \|Lx\| \geq \kappa \|x\|$ .
- (c)  $\liminf D_s^f(x_n) = 0$ .



Then  $x_n \rightarrow \bar{x}$ .

*Proof.* Set  $(\forall n \in \mathbb{N}) f_n = f$ . Then

$$(\forall n \in \mathbb{N}) \quad \begin{cases} f_n \in \mathcal{P}_1(f), \\ f \succcurlyeq f_n, \\ (1 + \eta_n)f_n \succcurlyeq f_{n+1}. \end{cases} \quad (3.52)$$

(i)(a): This is a corollary of Theorem 3.1(i).

(i)(b)–(i)(c): Firstly, the proof of Theorem 3.1(ii)(a)–(ii)(b) shows that  $\mathfrak{M}(x_n)_{n \in \mathbb{N}} \subset \text{int dom } f$ . Next, in view of Theorem 3.1(ii), it suffices to show that (3.4) holds. To this end, suppose that  $y_1$  and  $y_2$  are two weak sequential cluster points of  $(x_n)_{n \in \mathbb{N}}$  such that

$$(\langle y_1 - y_2, \nabla f(x_n) - \gamma_n L^* \nabla \psi(Lx_n) \rangle)_{n \in \mathbb{N}} \text{ converges.} \quad (3.53)$$

Then, there exist two strictly increasing sequences  $(k_n)_{n \in \mathbb{N}}$  and  $(l_n)_{n \in \mathbb{N}}$  in  $\mathbb{N}$  such that  $x_{k_n} \rightarrow y_1$  and  $x_{l_n} \rightarrow y_2$ . We derive from (3.50) and [27, Lemma 2.2.2] that there exists  $\theta \in [\varepsilon, \beta(1 - \varepsilon)]$  such that  $\gamma_n \rightarrow \theta$ . Since  $\nabla f$  and  $\nabla \psi$  are weakly sequentially continuous, after taking the limit in (3.53) along the subsequences  $(x_{k_n})_{n \in \mathbb{N}}$  and  $(x_{l_n})_{n \in \mathbb{N}}$ , respectively, we get

$$\langle y_1 - y_2, \nabla f(y_1) - \theta L^* \nabla \psi(Ly_1) \rangle = \langle y_1 - y_2, \nabla f(y_2) - \theta L^* \nabla \psi(Ly_2) \rangle. \quad (3.54)$$

Let us define

$$\begin{aligned} h: \mathcal{X} &\rightarrow ]-\infty, +\infty] \\ x &\mapsto \begin{cases} f(x) - \theta \psi(Lx), & \text{if } x \in \text{int dom } f; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \quad (3.55)$$

Then  $h$  is Gâteaux differentiable on  $\text{int dom } h = \text{int dom } f$  and (3.54) yields

$$\langle y_1 - y_2, \nabla h(y_1) - \nabla h(y_2) \rangle = 0. \quad (3.56)$$

On the other hand,

$$h - \varepsilon f = f - \theta \psi \circ L - \varepsilon f = (1 - \varepsilon)(f - \beta \psi \circ L) + (\beta(1 - \varepsilon) - \theta) \psi \circ L. \quad (3.57)$$

In turn, since  $f \succcurlyeq \beta \psi \circ L$  and  $\theta \leq \beta(1 - \varepsilon)$ , we obtain  $h \succcurlyeq \varepsilon f$ , and hence,

$$D^h(y_1, y_2) \geq \varepsilon D^f(y_1, y_2) \quad \text{and} \quad D^h(y_2, y_1) \geq \varepsilon D^f(y_2, y_1). \quad (3.58)$$

Therefore, (3.56) yields

$$\begin{aligned} 0 &= \langle y_1 - y_2, \nabla h(y_1) - \nabla h(y_2) \rangle \\ &= D^h(y_1, y_2) + D^h(y_2, y_1) \\ &\geq \varepsilon (D^f(y_1, y_2) + D^f(y_2, y_1)) \\ &= \varepsilon \langle y_1 - y_2, \nabla f(y_1) - \nabla f(y_2) \rangle. \end{aligned} \quad (3.59)$$

Suppose that  $y_1 \neq y_2$ . Since  $f|_{\text{int dom } f}$  is strictly convex,  $\nabla f$  is strictly monotone [28, Theorem 2.4.4(ii)], i.e.,

$$\langle y_1 - y_2, \nabla f(y_1) - \nabla f(y_2) \rangle > 0 \quad (3.60)$$

and we reach a contradiction.

(ii): The conclusions follow from (3.52) and Theorem 3.1(iii).  $\square$

**Remark 3.3** In Problem 1.1, suppose that  $L = \text{Id}$ . We rewrite algorithm (3.51) as follow

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \underset{x \in \mathcal{X}}{\operatorname{argmin}} \left( \varphi(x) + \langle x - x_n, \nabla \psi(x_n) \rangle + \psi(x_n) + \gamma_n^{-1} D^f(x, x_n) \right). \quad (3.61)$$

Another method to solve Problem 1.1 was proposed in [11]. In that method, instead of solving (3.61), the authors solve

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \underset{x \in \mathcal{X}}{\operatorname{argmin}} \left( \varphi(x) + \langle x - x_n, \nabla \psi(x_n) \rangle + \psi(x_n) + \gamma_n^{-1} \|x - x_n\|^p \right), \quad (3.62)$$

for some  $1 < p \leq 2$ . The weak convergence is established under the assumptions that Problem 1.1 admits a unique solution,  $\nabla \psi$  is  $(p-1)$ -Hölder continuous with constant  $\beta$ , and  $0 < \inf_{n \in \mathbb{N}} \gamma_n \leq \sup_{n \in \mathbb{N}} \gamma_n \leq (1-\delta)/\beta$ , where  $0 < \delta < 1$ . The high nonlinearity of the regularization in (3.62) compared to (3.61) makes the numerical implementation of this method difficult in general. Furthermore, since (3.62) yields

$$(\forall n \in \mathbb{N}) \quad 0 \in \partial \varphi(x_{n+1}) + \nabla \psi(x_n) + \gamma_n^{-1} \partial (\|x_{n+1} - x_n\|^p), \quad (3.63)$$

and since  $(\forall n \in \mathbb{N}) \partial (\|x_{n+1} - x_n\|^p)$  is not separable, this method is not a splitting method.

**Remark 3.4** We can reformulate Problem 1.1 as the following joint minimization problem

$$\underset{(x,y) \in V}{\operatorname{minimize}} \quad \varphi(x) + \psi(y), \quad (3.64)$$

where  $V = \operatorname{gra} L = \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y = Lx\}$ . This constrained problem is equivalent to the following unconstrained problem

$$\underset{(x,y) \in \mathcal{X} \times \mathcal{Y}}{\operatorname{minimize}} \quad \varphi(x) + \psi(y) + \iota_V(x, y). \quad (3.65)$$

In [9], a different coupling term between the variables  $x$  and  $y$  was considered and the problem considered there was

$$\underset{(x,y) \in \mathcal{X} \times \mathcal{Y}}{\operatorname{minimize}} \quad \varphi(x) + \psi(y) + D^f(x, y), \quad (3.66)$$

in the Euclidean spaces. Their method activates  $\varphi$  and  $\psi$  via their so-called left and right Bregman proximity operators alternatively (see also [7] for the projection setting). This method does not require the smoothness of  $\psi$  but it requires the computation of Bregman distance-based proximity operator of  $\psi$ .

Next, we provide a particular instance of Theorem 3.2 in finite-dimensional spaces.

**Corollary 3.5** *In the setting of Problem 1.1, suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  are finite-dimensional. Let  $f \in \Gamma_0(\mathcal{X})$  be a Legendre function such that  $\mathcal{S} \cap \text{int dom } f \neq \emptyset$ ,  $L(\text{int dom } f) \subset \text{int dom } \psi$ ,  $f \succcurlyeq \beta \psi \circ L$  for some  $\beta \in ]0, +\infty[$ , and  $\text{dom } f^*$  is open. Suppose that either  $f$  is cofinite or  $-L^*(\text{ran } \nabla \psi) \subset \text{dom } \varphi^*$ . Let  $\varepsilon \in ]0, \beta/(\beta+1)[$ , let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , and let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$  such that*

$$(\forall n \in \mathbb{N}) \quad \varepsilon \leq \gamma_n \leq \beta(1 - \varepsilon) \quad \text{and} \quad (1 + \eta_n)\gamma_n - \gamma_{n+1} \leq \beta\eta_n. \quad (3.67)$$

Furthermore, let  $x_0 \in \text{int dom } f$  and iterate

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \text{Prox}_{\gamma_n \varphi}^f(\nabla f(x_n) - \gamma_n L^* \nabla \psi(Lx_n)). \quad (3.68)$$

Then there exists  $\bar{x} \in \mathcal{S}$  such that  $x_n \rightarrow \bar{x}$ .

*Proof.* Since  $\text{dom } f^*$  is open, [5, Lemma 7.3(ix)] asserts that  $(\forall x \in \text{int dom } f) \ D^f(x, \cdot)$  is coercive. Hence, the claim follows from Theorem 3.2(i)(c).  $\square$

**Remark 3.6** We provide some special cases of Problem 1.1 and Theorem 3.2.

- (i) Let  $I$  and  $K$  be totally ordered countable index sets. In Problem 1.1, suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  are separable Hilbert spaces, and that  $\psi: y \mapsto \sum_{k \in K} |\langle y - r, y_k \rangle|^2 / 2$ , where  $r \in \mathcal{Y}$  and  $(y_k)_{k \in K}$  is a frame in  $\mathcal{Y}$ , i.e.,

$$(\exists (\mu, \nu) \in ]0, +\infty[^2)(\forall y \in \mathcal{Y}) \quad \mu \|y\|^2 \leq \sum_{k \in K} |\langle y, y_k \rangle|^2 \leq \nu \|y\|^2. \quad (3.69)$$

Then in Theorem 3.2, we can choose  $f: x \mapsto \sum_{i \in I} |\langle x, x_i \rangle|^2 / 2$ , where  $(x_i)_{i \in I}$  is a frame in  $\mathcal{X}$ , i.e.,

$$(\exists (\alpha, \beta) \in ]0, +\infty[^2)(\forall x \in \mathcal{X}) \quad \alpha \|x\|^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq \beta \|x\|^2. \quad (3.70)$$

It follows from [16, Corollary 1] that  $f$  and  $\psi$  are Legendre functions and that  $\nabla f$  and  $\nabla \psi$  are weakly sequentially continuous. Now let  $x$  and  $z$  be in  $\mathcal{X}$ . Then

$$\begin{aligned} D^\psi(Lx, Lz) &= \sum_{k \in K} |\langle Lx - Lz, y_k \rangle|^2 / 2 \leq \nu \|Lx - Lz\|^2 / 2 \\ &\leq \nu \|L\|^2 \|x - z\|^2 / 2 \leq \nu \|L\|^2 \alpha^{-1} \sum_{i \in I} |\langle x - z, x_i \rangle|^2 / 2 \\ &= \nu \|L\|^2 \alpha^{-1} D^f(x, z), \end{aligned} \quad (3.71)$$

which implies that  $f \succcurlyeq \alpha \nu^{-1} \|L\|^{-2} \psi \circ L$  and in addition,  $D^f(x, \cdot)$  is coercive.

- (ii) Let  $p$  and  $q$  be in  $]1, +\infty[$  and set  $p^* = p/(p-1)$  and  $q^* = q/(q-1)$ . In Problem 1.1, suppose that  $\mathcal{X} = \ell^p(\mathbb{N})$  and  $\mathcal{Y} = \ell^q(\mathbb{N})$ , that  $r \in \ell^q(\mathbb{N})$ , that  $\psi: y \mapsto \|y\|^q/q - \langle y - r, \|r\|^{q-2}r \rangle - \|r\|^q/q$ , and that there exists  $\kappa \in ]0, +\infty[$  such that  $(\forall x \in \ell^p(\mathbb{N})) \ \|Lx\| \geq \kappa \|x\|$ . It follows from [15, Theorem 4.7] that  $\ell^p(\mathbb{N})$  is uniformly convex and hence, strictly convex. Therefore,  $\psi$  is

strictly convex and supercoercive. The property of  $L$  implies that  $\psi \circ L$  is strictly convex and supercoercive, and  $\varphi + \psi \circ L$  is likewise. In turn, Problem 1.1 admits a unique solution. Let  $f \in \Gamma_0(\ell^p(\mathbb{N}))$  be a cofinite Legendre function such that  $f \succcurlyeq \beta \|L \cdot\|^p$  for some  $\beta \in ]0, +\infty[$ , let  $x \in \ell^p(\mathbb{N})$ , and set

$$\phi : [0, +\infty[ \rightarrow [0, +\infty] : t \mapsto \inf_{\|z-x\|=t} (\|x\|^p - p\langle x-z, \|z\|^{p-2}z \rangle - \|z\|^p). \quad (3.72)$$

Then

$$(\forall z \in \ell^p(\mathbb{N})) \quad D^f(x, z) \geq p\beta D^\psi(Lx, Lz) \geq \beta \phi(\|Lz - Lx\|). \quad (3.73)$$

If  $p \in [2, +\infty[$  then we derive from [12, Lemma 1.4.10] that

$$(\forall z \in \ell^p(\mathbb{N})) \quad \phi(\|Lz - Lx\|) \geq 2^{1-p} \|Lz - Lx\|^p \geq 2^{1-p} \kappa^p \|z - x\|^p. \quad (3.74)$$

Thus, it follows from (3.73) that

$$(\forall z \in \ell^p(\mathbb{N})) \quad D^f(x, z) \geq 2^{1-p} \kappa^p \beta \|z - x\|^p, \quad (3.75)$$

and  $D^f(x, \cdot)$  is therefore coercive. If  $p \in ]1, 2[$  then we derive from [12, Lemma 1.4.8] that

$$(\forall z \in \ell^p(\mathbb{N})) \quad \phi(\|Lz - Lx\|) \geq (\|Lz - Lx\| + \|Lz\|)^p - \|Lz\|^p - p\|Lz\|^{p-1} \|Lz - Lx\|, \quad (3.76)$$

and hence, (3.73) yields

$$(\forall z \in \ell^p(\mathbb{N})) \quad D^f(x, z) \geq \beta \left( (\|Lz\| + \|Lz - Lx\|)^p - \|Lz\|^p - p\|Lz\|^{p-1} \|Lz - Lx\| \right). \quad (3.77)$$

On the other hand, since

$$\lim_{\|Lz\| \rightarrow +\infty} \frac{(\|Lz\| + \|Lz - Lx\|)^p - \|Lz\|^p - p\|Lz\|^{p-1} \|Lz - Lx\|}{(2^p - 1 - p)\|Lz\|^p} = 1, \quad (3.78)$$

and since  $2^p - 1 - p > 0$ , it follows from (3.77) and the property of  $L$  that

$$\lim_{\|z\| \rightarrow +\infty} D^f(x, z) = +\infty, \quad (3.79)$$

and  $D^f(x, \cdot)$  is therefore coercive. Consequently, Theorem 3.2(i)(a) can be applied.

## 4 Application to multivariate minimization

We propose a variant of the forward-backward algorithm to solve the following multivariate minimization problem.

**Problem 4.1** Let  $m$  and  $p$  be strictly positive integers, let  $(\mathcal{X}_i)_{1 \leq i \leq m}$  and  $(\mathcal{Y}_k)_{1 \leq k \leq p}$  be reflexive real Banach spaces. For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, p\}$ , let  $\varphi_i \in \Gamma_0(\mathcal{X}_i)$ , let  $\psi_k \in \Gamma_0(\mathcal{Y}_k)$  be Gâteaux differentiable on  $\text{int dom } \psi_k \neq \emptyset$ , and let  $L_{ik} : \mathcal{X}_i \rightarrow \mathcal{Y}_k$  be linear and bounded. The problem is to

$$\underset{x_1 \in \mathcal{X}_1, \dots, x_m \in \mathcal{X}_m}{\text{minimize}} \quad \sum_{i=1}^m \varphi_i(x_i) + \sum_{k=1}^p \psi_k \left( \sum_{i=1}^m L_{ik} x_i \right). \quad (4.1)$$

Denote by  $\mathcal{S}$  the set of solutions to (4.1).

We derive from Theorem 3.2 the following result.

**Proposition 4.2** Consider the setting of Problem 4.1. For every  $k \in \{1, \dots, p\}$ , suppose that there exists  $\sigma_k \in ]0, +\infty[$  such that for every  $(y_{ik})_{1 \leq i \leq m} \in \text{int dom } \psi_k$  and every  $(v_{ik})_{1 \leq i \leq m} \in \text{int dom } \psi_k$  satisfying  $\sum_{i=1}^m y_{ik} \in \text{int dom } \psi_k$  and  $\sum_{i=1}^m v_{ik} \in \text{int dom } \psi_k$ , one has

$$D^{\psi_k} \left( \sum_{i=1}^m y_{ik}, \sum_{i=1}^m v_{ik} \right) \leq \sigma_k \sum_{i=1}^m D^{\psi_k}(y_{ik}, v_{ik}). \quad (4.2)$$

For every  $i \in \{1, \dots, m\}$ , let  $f_i \in \Gamma_0(\mathcal{X}_i)$  be a Legendre function such that  $(\forall x_i \in \text{int dom } f_i) D^{f_i}(x_i, \cdot)$  is coercive. For every  $k \in \{1, \dots, p\}$ , suppose that  $\sum_{i=1}^m L_{ik}(\text{int dom } f_i) \subset \text{int dom } \psi_k$ , that, for every  $i \in \{1, \dots, m\}$ , there exists  $\beta_{ik} \in ]0, +\infty[$  such that  $f_i \succ \beta_{ik} \psi_k \circ L_{ik}$ , and set  $\beta_k = \min_{1 \leq i \leq m} \beta_{ik}$ . In addition, suppose that  $\mathcal{S} \cap \bigtimes_{i=1}^m \text{int dom } f_i \neq \emptyset$  and that either  $(\forall i \in \{1, \dots, m\}) f_i$  is cofinite or  $(\forall i \in \{1, \dots, m\}) \varphi_i$  is cofinite. Let  $\varepsilon \in ]0, 1/(1 + \sum_{k=1}^p \sigma_k \beta_k^{-1})[$ , let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , and let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$  such that

$$(\forall n \in \mathbb{N}) \quad \varepsilon \leq \gamma_n \leq \frac{1 - \varepsilon}{\sum_{k=1}^p \sigma_k \beta_k^{-1}} \quad \text{and} \quad (1 + \eta_n) \gamma_n - \gamma_{n+1} \leq \frac{\eta_n}{\sum_{k=1}^p \sigma_k \beta_k^{-1}}. \quad (4.3)$$

Furthermore, let  $(x_{i,0})_{1 \leq i \leq m} \in \bigtimes_{i=1}^m \text{int dom } f_i$  and iterate

$$\begin{aligned} & \text{for } n = 0, 1, \dots \\ & \left[ \begin{array}{l} \text{for } i = 1, \dots, m \\ \quad x_{i,n+1} = \text{Prox}_{\gamma_n \varphi_i}^{f_i} (\nabla f_i(x_{i,n}) - \gamma_n \sum_{k=1}^p L_{ik}^* \nabla \psi_k (\sum_{j=1}^m L_{jk} x_{j,n})) \end{array} \right. \end{aligned} \quad (4.4)$$

Then there exists  $(\bar{x}_i)_{1 \leq i \leq m} \in \mathcal{S}$  such that the following hold.

- (i) Suppose that  $\mathcal{S} \cap \bigtimes_{i=1}^m \overline{\text{dom } f_i}$  is a singleton. Then  $(\forall i \in \{1, \dots, m\}) x_{i,n} \rightarrow \bar{x}_i$ .
- (ii) For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, p\}$ , suppose that  $\nabla f_i$  and  $\nabla \psi_k$  are weakly sequentially continuous, and that one of the following holds.
  - (a)  $\text{dom } \varphi_i \subset \text{int dom } f_i$ .
  - (b)  $\text{dom } f_i^*$  is open and  $\nabla f_i^*$  is weakly sequentially continuous.

Then  $(\forall i \in \{1, \dots, m\}) x_{i,n} \rightarrow \bar{x}_i$ .

*Proof.* Denote by  $\mathcal{X}$  and  $\mathcal{Y}$  the standard vector product spaces  $\bigtimes_{i=1}^m \mathcal{X}_i$  and  $\bigtimes_{k=1}^p \mathcal{Y}_k$  equipped with the norms  $x = (x_i)_{1 \leq i \leq m} \mapsto \sqrt{\sum_{i=1}^m \|x_i\|^2}$  and  $y = (y_k)_{1 \leq k \leq p} \mapsto \sqrt{\sum_{k=1}^p \|y_k\|^2}$ , respectively. Then  $\mathcal{X}^*$  is the vector product space  $\bigtimes_{i=1}^m \mathcal{X}_i^*$  equipped with the norm  $x^* \mapsto \sqrt{\sum_{i=1}^m \|x_i^*\|^2}$  and  $\mathcal{Y}^*$  is the vector product space  $\bigtimes_{k=1}^p \mathcal{Y}_k^*$  equipped with the norm  $y^* \mapsto \sqrt{\sum_{k=1}^p \|y_k^*\|^2}$ . Let us introduce the functions and operator

$$\begin{cases} \varphi: \mathcal{X} \rightarrow ]-\infty, +\infty] : x \mapsto \sum_{i=1}^m \varphi_i(x_i) \\ f: \mathcal{X} \rightarrow ]-\infty, +\infty] : x \mapsto \sum_{i=1}^m f_i(x_i) \\ \psi: \mathcal{Y} \rightarrow ]-\infty, +\infty] : y \mapsto \sum_{k=1}^p \psi_k(y_k) \\ L: \mathcal{X} \rightarrow \mathcal{Y} : x \mapsto (\sum_{i=1}^m L_{ik} x_i)_{1 \leq k \leq p}. \end{cases} \quad (4.5)$$

Then  $\psi$  is Gâteaux differentiable on  $\text{int dom } \psi = \bigtimes_{k=1}^p \text{int dom } \psi_k$  and Problem 4.1 is a special case of Problem 1.1. Since (4.5) yields  $\text{dom } f^* = \bigtimes_{i=1}^m \text{dom } f_i^*$  and  $\text{dom } \varphi^* = \bigtimes_{i=1}^m \text{dom } \varphi_i^*$ , we deduce from our assumptions that either  $f$  is cofinite or  $\varphi$  is cofinite. As in (2.18) and (2.19),  $f$  is a Legendre function and  $\text{dom } \varphi \cap \text{int dom } f \neq \emptyset$ . In addition,

$$L(\text{int dom } f) = \bigtimes_{k=1}^p \sum_{i=1}^m L_{ki}(\text{int dom } f_i) \subset \bigtimes_{k=1}^p \text{int dom } \psi_k = \text{int dom } \psi. \quad (4.6)$$

Now let  $x \in \text{int dom } f$ . First, to show that  $D^f(x, \cdot)$  is coercive, we fix  $\rho \in \mathbb{R}$ . On the one hand,

$$\{z = (z_i)_{1 \leq i \leq m} \in \mathcal{X} \mid D^f(x, z) \leq \rho\} \subset \bigtimes_{i=1}^m \{z_i \in \mathcal{X}_i \mid D^{f_i}(x_i, z_i) \leq \rho\}. \quad (4.7)$$

On the other hand, for every  $i \in \{1, \dots, m\}$ , since  $D^{f_i}(x_i, \cdot)$  is coercive, we deduce that

$$\{z_i \in \mathcal{X}_i \mid D^{f_i}(x_i, z_i) \leq \rho\} \text{ is bounded.} \quad (4.8)$$

Hence (4.7) implies that  $\{z \in \mathcal{X} \mid D^f(x, z) \leq \rho\}$  is bounded and  $D^f(x, \cdot)$  is therefore coercive. Next, set  $\beta = 1 / \sum_{k=1}^p \sigma_k \beta_k^{-1}$ . We shall show that  $f \succcurlyeq \beta \psi \circ L$ . To this end, fix  $z = (z_i)_{1 \leq i \leq m} \in \text{int dom } f$ . We have

$$\begin{aligned} D^\psi(Lx, Lz) &= \sum_{k=1}^p D^{\psi_k} \left( \sum_{i=1}^m L_{ik} x_i, \sum_{i=1}^m L_{ik} z_i \right) \\ &\leq \sum_{k=1}^p \sum_{i=1}^m \sigma_k D^{\psi_k}(L_{ik} x_i, L_{ik} z_i) \\ &\leq \sum_{k=1}^p \sum_{i=1}^m \sigma_k \beta_{ik}^{-1} D^{f_i}(x_i, z_i) \\ &\leq \sum_{k=1}^p \sigma_k \beta_k^{-1} D^f(x, z). \end{aligned} \quad (4.9)$$

Now let us set  $(\forall n \in \mathbb{N}) x_n = (x_{i,n})_{1 \leq i \leq m}$ . By virtue of Proposition 2.8, (4.4) is a particular case of (3.51).

(i): Since  $\mathcal{S} \cap \overline{\text{dom } f}$  is a singleton, the claim follows from Theorem 3.2(i)(a).

(ii): Our assumptions on  $(f_i)_{1 \leq i \leq m}$  and  $(\psi_k)_{1 \leq k \leq p}$  imply that  $\nabla f$  and  $\nabla \psi$  are weakly sequentially continuous.

(ii)(a): Since  $\mathcal{S} \subset \bigtimes_{i=1}^m \text{dom } \varphi_i \subset \bigtimes_{i=1}^m \text{int dom } f_i = \text{int dom } f$ , the claim follows from Theorem 3.2(i)(b).

(ii)(b): Since, for every  $i \in \{1, \dots, m\}$ ,  $\text{dom } f_i^*$  is open and  $\nabla f_i^*$  is weakly sequentially continuous, we deduce that  $\text{dom } f^*$  is open and  $\nabla f^*$  is weakly sequentially continuous. The assertion therefore follows from Theorem 3.2(i)(c).  $\square$

**Example 4.3** In Problem 4.1, suppose that  $m = 1$ , that  $\mathcal{X}_1$  and  $(\mathcal{Y}_k)_{1 \leq k \leq p}$  are Hilbert spaces, and that, for every  $k \in \{1, \dots, p\}$ ,  $\varphi_k = \omega_k \|\cdot - r_k\|^2/2$ , where  $(\omega_k)_{1 \leq k \leq p} \in ]0, +\infty[^p$  and let  $(r_k)_{1 \leq k \leq p} \in \times_{k=1}^p \mathcal{Y}_k$ . Then the weak convergence result in [17, Proposition 6.3] without errors is a particular instance of Proposition 4.2 with  $f_1 = \|\cdot\|^2/2$ .

**Example 4.4** Let  $m$  and  $p$  be strictly positive integers. For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, p\}$ , let  $\omega_{ik} \in ]0, +\infty[$ , let  $\varrho_k \in ]0, +\infty[$ , and let  $\varphi_i \in \Gamma_0(\mathbb{R})$  be cofinite. The problem is to

$$\underset{(\xi_1, \dots, \xi_m) \in ]0, +\infty[^m}{\text{minimize}} \quad \sum_{i=1}^m \varphi_i(\xi_i) + \sum_{k=1}^p \left( -\ln \frac{\sum_{i=1}^m \omega_{ik} \xi_i}{\varrho_k} + \frac{\sum_{i=1}^m \omega_{ik} \xi_i}{\varrho_k} - 1 \right). \quad (4.10)$$

Denote by  $\mathcal{S}$  the set of solutions to (4.10) and suppose that  $\mathcal{S} \cap ]0, +\infty[^m \neq \emptyset$ . Let

$$\vartheta: \mathbb{R} \rightarrow ]-\infty, +\infty] : \xi \mapsto \begin{cases} -\ln \xi, & \text{if } \xi > 0; \\ +\infty, & \text{otherwise} \end{cases} \quad (4.11)$$

be Burg entropy, let  $\varepsilon \in ]0, 1/(1+p)[$ , let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , and let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$  such that

$$(\forall n \in \mathbb{N}) \quad \varepsilon \leq \gamma_n \leq p^{-1}(1 - \varepsilon) \quad \text{and} \quad (1 + \eta_n)\gamma_n - \gamma_{n+1} \leq p^{-1}\eta_n. \quad (4.12)$$

Let  $(\xi_{i,0})_{1 \leq i \leq m} \in ]0, +\infty[^m$  and iterate

$$\begin{aligned} & \text{for } n = 0, 1, \dots \\ & \left[ \begin{array}{l} \text{for } i = 1, \dots, m \\ \xi_{i,n+1} = \text{Prox}_{\gamma_n \varphi_i}^{\vartheta} \left( \frac{-1}{\xi_{i,n}} - \gamma_n \sum_{k=1}^p \omega_{ik} \left( \frac{-1}{\sum_{j=1}^m \omega_{jk} \xi_{j,n}} + \frac{1}{\varrho_k} \right) \right) \end{array} \right] \end{aligned} \quad (4.13)$$

Then there exists  $(\bar{\xi}_i)_{1 \leq i \leq m} \in \mathcal{S}$  such that  $(\forall i \in \{1, \dots, m\}) \quad \xi_{i,n} \rightarrow \bar{\xi}_i$ .

*Proof.* For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, p\}$ , let us set  $\mathcal{X}_i = \mathbb{R}$ ,  $\mathcal{Y}_k = \mathbb{R}$ ,  $\psi_k = D^{\vartheta}(\cdot, \varrho_k)$ , and  $L_{ik}: \xi_i \mapsto \omega_{ik} \xi_i$ . Then (4.10) is a particular case of (4.1). Since  $\psi$  is not differentiable on  $\mathbb{R}^p$ , the standard forward-backward algorithm is inapplicable. We show that the problem can be solved by using Proposition 4.2. First, let  $(\xi_i)_{1 \leq i \leq m}$  and  $(\eta_i)_{1 \leq i \leq m}$  be in  $]0, +\infty[^m$ , and consider

$$\phi: \mathbb{R} \rightarrow ]-\infty, +\infty] : \xi \mapsto \begin{cases} -\ln \xi + \xi - 1, & \text{if } \xi \in ]0, +\infty[; \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.14)$$

We see that  $\phi$  is convex and positive. Thus,

$$\phi \left( \frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i} \right) = \phi \left( \sum_{i=1}^m \frac{\eta_i}{\sum_{j=1}^m \eta_j} \frac{\xi_i}{\eta_i} \right) \leq \sum_{i=1}^m \frac{\eta_i}{\sum_{j=1}^m \eta_j} \phi \left( \frac{\xi_i}{\eta_i} \right) \leq \sum_{i=1}^m \phi \left( \frac{\xi_i}{\eta_i} \right), \quad (4.15)$$

and hence,

$$-\ln \frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i} + \frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i} - 1 \leq \sum_{i=1}^m \left( -\ln \frac{\xi_i}{\eta_i} + \frac{\xi_i}{\eta_i} - 1 \right). \quad (4.16)$$

In turn,

$$D^\vartheta \left( \sum_{i=1}^m \xi_i, \sum_{i=1}^m \eta_i \right) \leq \sum_{i=1}^m D^\vartheta(\xi_i, \eta_i). \quad (4.17)$$

This shows that (4.2) is satisfied with  $(\forall k \in \{1, \dots, p\}) \sigma_k = 1$ . Next, let us set  $(\forall i \in \{1, \dots, m\}) f_i = \vartheta$ . Fix  $i \in \{1, \dots, m\}$  and  $k \in \{1, \dots, p\}$ , and let  $\xi_i$  and  $\eta_i$  be in  $]0, +\infty[$ . Then

$$D^{\psi_k}(L_{ik}\xi_i, L_{ik}\eta_i) = D^\vartheta(\omega_{ik}\xi_i, \omega_{ik}\eta_i) = D^\vartheta(\xi_i, \eta_i) = D^{f_i}(\xi_i, \eta_i), \quad (4.18)$$

which implies that  $f_i \succcurlyeq \psi_k \circ L_{ik}$ . In addition, since  $\text{dom } f_i^* = ]-\infty, 0[$  is open, [5, Lemma 7.3(ix)] asserts that  $D^{f_i}(\xi_i, \cdot)$  is coercive. We therefore deduce the convergence result from Proposition 4.2(ii)(b).  $\square$

**Example 4.5** Let  $m$  and  $p$  be strictly positive integers. For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, p\}$ , let  $\omega_{ik} \in ]0, +\infty[$ , let  $\varrho_k \in ]0, +\infty[$ , and let  $\varphi_i \in \Gamma_0(\mathbb{R})$ . The problem is to

$$\underset{(\xi_1, \dots, \xi_m) \in [0, +\infty[^m}{\text{minimize}} \quad \sum_{i=1}^m \varphi_i(\xi_i) + \sum_{k=1}^p \left( \left( \sum_{i=1}^m \omega_{ik} \xi_i \right) \ln \frac{\sum_{i=1}^m \omega_{ik} \xi_i}{\varrho_k} - \sum_{i=1}^m \omega_{ik} \xi_i + \varrho_k \right). \quad (4.19)$$

Denote by  $\mathcal{S}$  the set of solutions to (4.19) and suppose that  $\mathcal{S} \cap ]0, +\infty[^m \neq \emptyset$ . Let

$$\vartheta: \mathbb{R} \rightarrow ]-\infty, +\infty] : \xi \mapsto \begin{cases} \xi \ln \xi - \xi, & \text{if } \xi \in ]0, +\infty[; \\ 0, & \text{if } \xi = 0; \\ +\infty, & \text{otherwise} \end{cases} \quad (4.20)$$

be Boltzmann-Shannon entropy, let  $\beta = \max_{1 \leq k \leq p} \max_{1 \leq i \leq m} \omega_{ik}$ , let  $\varepsilon \in ]0, 1/(1 + \beta)[$ , let  $(\eta_n)_{n \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ , and let  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$  such that

$$(\forall n \in \mathbb{N}) \quad \varepsilon \leq \gamma_n \leq (p\beta)^{-1}(1 - \varepsilon) \quad \text{and} \quad (1 + \eta_n)\gamma_n - \gamma_{n+1} \leq (p\beta)^{-1}\eta_n. \quad (4.21)$$

Let  $(\xi_{i,0})_{1 \leq i \leq m} \in ]0, +\infty[^m$  and iterate

$$\begin{aligned} & \text{for } n = 0, 1, \dots \\ & \quad \left[ \begin{array}{l} \text{for } i = 1, \dots, m \\ \xi_{i,n+1} = \text{Prox}_{\gamma_n \varphi_i}^\vartheta \left( \ln \xi_{i,n} - \gamma_n \sum_{k=1}^p \omega_{ik} \left( \ln \left( \sum_{j=1}^m \omega_{jk} \xi_{j,n} \right) - \ln \varrho_k \right) \right) \end{array} \right. \end{aligned} \quad (4.22)$$

Then there exists  $(\bar{\xi}_i)_{1 \leq i \leq m} \in \mathcal{S}$  such that  $(\forall i \in \{1, \dots, m\}) \xi_{i,n} \rightarrow \bar{\xi}_i$ .

*Proof.* For every  $i \in \{1, \dots, m\}$  and every  $k \in \{1, \dots, p\}$ , let us set  $\mathcal{X}_i = \mathbb{R}$ ,  $\mathcal{Y}_k = \mathbb{R}$ ,  $\psi_k = D^\vartheta(\cdot, \varrho_k)$ , and  $L_{ik}: \xi_i \mapsto \omega_{ik}\xi_i$ . Then (4.19) is a particular case of (4.1). We cannot apply the standard forward-backward algorithm here since  $\psi$  is not differentiable on  $\mathbb{R}^p$ . We shall verify the assumptions of Proposition 4.2. First, let  $(\xi_i)_{1 \leq i \leq m}$  and  $(\eta_i)_{1 \leq i \leq m}$  be in  $]0, +\infty[^m$ . Since

$$\phi: \mathbb{R} \rightarrow ]-\infty, +\infty] : \xi \mapsto \begin{cases} \xi \ln \xi, & \text{if } \xi \in ]0, +\infty[; \\ 0, & \text{if } \xi = 0; \\ +\infty, & \text{otherwise} \end{cases} \quad (4.23)$$



is convex, we have

$$\phi\left(\frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i}\right) = \phi\left(\sum_{i=1}^m \frac{\eta_i}{\sum_{j=1}^m \eta_j} \frac{\xi_i}{\eta_i}\right) \leq \sum_{i=1}^m \frac{\eta_i}{\sum_{j=1}^m \eta_j} \phi\left(\frac{\xi_i}{\eta_i}\right), \quad (4.24)$$

and hence,

$$\frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i} \ln \frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i} \leq \sum_{i=1}^m \frac{\eta_i}{\sum_{j=1}^m \eta_j} \frac{\xi_i}{\eta_i} \ln \frac{\xi_i}{\eta_i} = \frac{\sum_{i=1}^m \xi_i \ln \frac{\xi_i}{\eta_i}}{\sum_{i=1}^m \eta_i}. \quad (4.25)$$

In turn,

$$\left(\sum_{i=1}^m \xi_i\right) \ln \frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i} \leq \sum_{i=1}^m \xi_i \ln \frac{\xi_i}{\eta_i}, \quad (4.26)$$

which implies that

$$\begin{aligned} D^\vartheta\left(\sum_{i=1}^m \xi_i, \sum_{i=1}^m \eta_i\right) &= \left(\sum_{i=1}^m \xi_i\right) \ln \frac{\sum_{i=1}^m \xi_i}{\sum_{i=1}^m \eta_i} - \sum_{i=1}^m \xi_i + \sum_{i=1}^m \eta_i \\ &\leq \sum_{i=1}^m \left(\xi_i \ln \frac{\xi_i}{\eta_i} - \xi_i + \eta_i\right) \\ &= \sum_{i=1}^m D^\vartheta(\xi_i, \eta_i). \end{aligned} \quad (4.27)$$

This shows that (4.2) is satisfied with  $(\forall k \in \{1, \dots, p\}) \sigma_k = 1$ . Next, let us set  $(\forall i \in \{1, \dots, m\}) f_i = \vartheta$ . Fix  $i \in \{1, \dots, m\}$  and  $k \in \{1, \dots, p\}$ , and let  $\xi_i$  and  $\eta_i$  be in  $]0, +\infty[$ . Then

$$D^{\psi_k}(L_{ik}\xi_i, L_{ik}\eta_i) = D^\vartheta(\omega_{ik}\xi_i, \omega_{ik}\eta_i) = \omega_{ik} D^\vartheta(\xi_i, \eta_i) \leq \beta D^\vartheta(\xi_i, \eta_i), \quad (4.28)$$

which implies that  $f_i \succ \beta^{-1}\psi_k \circ L_{ik}$ . In addition, since  $f_i$  is supercoercive,  $f_i$  is cofinite and [5, Lemma 7.3(viii)] asserts that  $D^{f_i}(\xi_i, \cdot)$  is coercive. Therefore, the claim follows from Proposition 4.2(ii)(b).  $\square$

**Remark 4.6** The Bregman distance associated with Burg entropy, i.e., the Itakura-Saito divergence, is used in linear regression [3, Section 3]. The Bregman distance associated with Boltzmann-Shannon entropy, i.e., the Kullback-Leibler divergence, is used in information theory [3, Section 3] and image processing [13].

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